PERIODIC QUADRATIC TRANSFORMATIONS IN THE PLANE.

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In his prize essay on periodic transformations, Kantor * mentions one class in which each power of a quadratic transformation is quadratic, unless it is periodic, in which case each power except the identity is a quadratic transformation.† No equations are given, and but few of the properties of the transformation are derived. It is probably for this reason that an important category of quadratic transformations is omitted from Kantor's list, and consequently whole groups "with two points" should be added to Wiman's classification.‡ There is no error in the latter paper, but the results of Kantor's investigations are assumed to be correct. By applying the same methods, as used by Wiman, upon the omitted forms a complete list can be obtained. It is the purpose of this paper to obtain the equation, define the system of fundamental elements, and discuss some of the properties of these transformations, and to show the existence of new ones not heretofore considered.||

1. Let $x_i' = \varphi_i(x_1, x_2, x_3) = \varphi_i(x)$ $[i = 1, 2, 3]$ define a birational quadratic transformation T of period 3 in a ternary field. Let A, B, C be the fundamental points in $[x]$, and A', B', C' those in $[x']$ such that the image of A is $B'C'$, etc. Finally, suppose that x_i, x_i' are referred to the same triangle of reference and the same unit point. An arbitrary line $c_1(x)$ will go into a conic $c_2(x')$ through A', B', C'. This conic, regarded as in $[x]$, will have for image in $[x']$ a rational quartic having double points at A', B', C'. On the other hand, from the relation $T^3 = 1$ or $T^2 = T^{-1}$ it follows that the line c_1, regarded as in $[x']$, will go into a conic circumscribing the triangle ABC. This conic must be identical with the quartic with double points at A', B', C', after extraneous factors have been removed. This condition

* S. Kantor: Premiers fondements pour une théorie des transformations periodiques univoques; mémoire couronnée par l'académie des sciences physiques et mathématiques de Naples dans le concours pour 1883, Atti della R. accademia delle scienze di Napoli, ser. 2, vols. 3 and 4 (1891), pp. 1–356.

† L. c., § 30.

‡ A. Wiman: Ueber die endlichen Gruppen von birationalen Transformationen in der Ebene, Mathematische Annalen, vol. 48 (1895), pp. 195–235.

|| For the definitions of the various words used throughout the paper as well as for a systematic treatment of the elementary properties of these transformations, see Miss Scott's Introduction to modern analytic geometry (1892), pp. 218–225, or Doehlemann, Geometrische Transformationen, Zweiter Teil (1908), pp. 1–23.

necessitates that *the two triangles ABC, $A'B'C'$ have two and only two vertices in common,* and that the third vertex of each can not lie on any side of the other, since the passage of a curve through a fundamental point is the necessary and sufficient condition that its image should contain a fundamental straight line as a factor.

We shall first assume that the points A, B, C are distinct. The cases in which two or all three approach coincidence will be considered later. Several cases may now arise, but all the others may be derived from one by multiplying a certain quadratic transformation by an appropriate cyclic collineation. Assume $B \equiv B'$, $C \equiv C'$.

2. The image of the point A is, by hypothesis, the line $B'C' = BC$. The image of BC is, by hypothesis, the point A'. Since $T^3 = 1$, it follows that the image of A', regarded as a point of $[x]$, is A, regarded as a point of $[x']$.

Since no three of these points are collinear and no two approach coincidence, we may take, without loss of generality

$$A \equiv (1, 0, 0), \quad B \equiv (0, 1, 0), \quad C \equiv (0, 0, 1), \quad A' \equiv (1, 1, 1).$$

Since by T a straight line in $[x]$ goes into a conic in $[x']$ through A', $B' \equiv B$, $C' \equiv C$, we may write

$$\rho x_1 = a(x_1'^2 - x_2'x_3') + bx_2'(x_1' - x_3') + cx_3'(x_1' - x_2'),$$
$$\rho x_2 = a'(x_1'^2 - x_2'x_3') + b'x_2'(x_1' - x_3') + c'x_3'(x_1' - x_2'),$$
$$\rho x_3 = a''(x_1'^2 - x_2'x_3') + b''x_2'(x_1' - x_3') + c''x_3'(x_1' - x_2').$$

Any line $m_3x_2 - m_2x_3 = 0$ through A goes into a conic having $x_1' = 0$ as factor, for every value of $m_2 : m_3$, hence

$$a' + b' + c' = 0,$$
$$a'' + b'' + c'' = 0.$$

In the same way, since a line $m_3x_1 - m_1x_3 = 0$ through B goes into a conic having $x_1' - x_2' = 0$ as factor, we obtain the further relations

$$a + b = 0, \quad a'' + b'' = 0.$$

A line $m_2x_1 - m_1x_2 = 0$ goes into a conic having $x_1' - x_3' = 0$ as factor, from which

$$a + c = 0, \quad a' + c' = 0.$$

The point $(1, 0, 0)$ in $[x']$ has $(1, 1, 1)$ for its image in $[x]$, hence

$$a = a' = a''.$$

These eight equations of condition are just sufficient to determine the coefficients in the equations of the transformation. The result is:

$$T:\quad \begin{aligned} \sigma x_1' &= x_2x_3, \\ \sigma x_2' &= x_3(x_2 - x_1), \\ \sigma x_3' &= x_2(x_3 - x_1), \end{aligned} \qquad T^{-1}:\quad \begin{aligned} \rho x_1 &= (x_1' - x_2')(x_1' - x_3'), \\ \rho x_2 &= x_1'(x_1' - x_3'), \\ \rho x_3 &= x_1'(x_1' - x_2'). \end{aligned}$$

3. The points of the plane are arranged in triads. If the points of a triad are collinear, the locus containing the triads will be invariant under the transformation. The equation of the locus is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_2x_3 & x_3(x_2 - x_1) & x_2(x_3 - x_1) \\ (x_1 - x_2)(x_1 - x_3) & x_1(x_1 - x_3) & x_1(x_1 - x_2) \end{vmatrix} = 0;$$

it consists of the five lines

$$x_2 - x_3 = 0, \quad x_1 - \theta x_3 = 0, \quad x_1 + \theta^2 x_3 = 0, \quad x_1 - \theta x_2 = 0, \quad x_1 + \theta^2 x_2 = 0 \quad [\theta^3 = -1].$$

Of these, the first, second, and fourth meet at $D_1 \equiv (\theta, 1, 1)$, while the first, third, and fifth meet at $D_2 \equiv (-\theta^2, 1, 1)$. Thus the invariant line $x_2 - x_3 = 0$ contains two of the self-corresponding points. The others are $D_3 \equiv (\theta, \theta^2, -1)$, the intersection of $x_1 - \theta x_3 = 0$, $x_1 + \theta^2 x_2 = 0$ and $D_4 \equiv (\theta, 1, \theta^2)$, the intersection of $x_1 - \theta x_2 = 0, x_1 + \theta^2 x_3 = 0$.

4. The pencil of lines through C remains invariant, a triad being formed by the lines

$$x_2 = kx_1, \quad x_2 = \frac{x_1}{1-k}, \quad x_2 = \frac{k-1}{k}x_1.$$

Similarly for the pencil through B.

5. In the preceding equations of condition among the coefficients, the first six were obtained independently of the periodicity. If $T^n = 1$, the last two equations are replaced by $A'T^{n-2}A$. All the coefficients are linear functions of a, a', a''; of these, a', a'' enter symmetrically. The equations now become

$$T:\quad \begin{aligned} \sigma x_1' &= ax_2x_3, \\ \sigma x_2' &= x_3(ax_2 - a'x_1), \\ \sigma x_3' &= x_2(ax_3 - a''x_1), \end{aligned} \qquad T^{-1}:\quad \begin{aligned} \rho x_1 &= a(x_1' - x_2')(x_1' - x_3'), \\ \rho x_2 &= a'x_1'(x_1' - x_3'), \\ \rho x_3 &= a''x_1'(x_1' - x_2'). \end{aligned}$$

If we put $a' = a'' = 1$, the following properties are now apparent. The pencils of straight lines through B and through C are invariant. The transformation is a collineation for each of these pencils, leaving two

lines invariant. Those of B are $x_1 - m_1x_3 = 0$, $x_1 - m_2x_3 = 0$, m_1, m_2 being defined by

$$m^2 - am + a = 0.$$

The lines $x_1 - m_1x_2 = 0$, $x_1 - m_2x_2 = 0$ are invariant through C. The line $AA' \equiv x_2 - x_3 = 0$ is invariant and contains all the $n - 2$ images of A'.

The lines $x_1 - m_1x_2 = 0$, $x_1 - m_1x_3 = 0$ intersect in D_1, $x_1 - m_2x_2 = 0$, $x_1 - m_2x_3 = 0$ intersect in D_2; D_1, D_2 both lie on $x_2 - x_3 = 0$. The lines $x_1 - m_1x_3 = 0$, $x_1 - m_2x_2 = 0$ intersect in D_3; $x_1 - m_2x_3 = 0$, $x_1 - m_1x_2 = 0$ intersect in D_4. The points

$$D_1 \equiv (m_1m_2,\ m_2,\ m_2), \quad D_2 \equiv (m_1m_2,\ m_1,\ m_1),$$

$$D_3 \equiv (m_1m_2,\ m_1,\ m_2), \quad D_4 \equiv (m_1m_2,\ m_2,\ m_1)$$

are the only invariant points of the transformation T. Every conic $k_1(x_1 - m_1x_3)(x_1 - m_2x_2) - k_2(x_1 - m_1x_2)(x_1 - m_2x_3) = 0$ of the pencil through B, C, D_1, D_2 remains invariant. Thus, *any point P and all its images under T lie on the same conic of the pencil, when $a' = a''$.*

6. The value of a can be readily determined when n is given. If $x_1^{(k)}$, $x_2^{(k)}$, $x_3^{(k)}$ are the coordinates of the kth image of (x_1, x_2, x_3) when operated upon by T, then the point $A' \equiv (1, 1, 1)$ becomes

$$x_1^{(k)} = \beta' \cdot \beta'', \quad x_1^{(k+1)} = \gamma' \cdot \gamma'',$$

$$x_2^{(k)} = \beta'' \cdot \gamma', \quad x_2^{(k+1)} = \gamma'' \cdot \delta',$$

$$x_3^{(k)} = \beta' \cdot \gamma'', \quad x_3^{(k+1)} = \gamma' \cdot \delta'',$$

wherein $\delta' = \gamma' - a'\beta'$, $\delta'' = \gamma'' - a''\beta''$, each function l'' being the same function of a and a'', as l' is of a and a'. If

$$k = 2,$$

then
$$\beta' = a - a',$$
$$\gamma' = a - 2a';$$

putting $a' = a'' = 1$, the following formulas are obtained:

$$\begin{array}{rll} a = 1 & \text{for} & n = 3 \\ a = 2 & & n = 4 \\ a^2 - 3a + 1 = 0 & & n = 5 \\ a^2 - 4a + 3 = 0 & & n = 6 \\ a^3 - 5a^2 + 6a - 1 = 0 & & n = 7 \\ a^3 - 6a^2 + 10a - 4 = 0 & & n = 8 \\ \cdots\cdots & & \cdots \end{array}$$

For any composite value of n, one root a of the associated equation belongs also to the equation associated with that factor. All the roots of the equation are real.

7. The fundamental triangles all have the vertices B, C in common; the third vertex of each always lies on the invariant line $x_2 - x_3 = 0$.

$$\begin{array}{llll} \text{For } n = 3, & T \text{ has } ABC \text{ in } [x], & A'BC \text{ in } [x'], \\ & T^2 \text{ has } A'BC \text{ in } [x], & ABC \text{ in } [x']. \\ \text{For } n = 4, & T \text{ has } ABC \text{ in } [x], & A'BC \text{ in } [x'], \\ & T^2 \text{ has } A_1'BC \text{ in } [x], & A_1'BC \text{ in } [x'], \\ & T^3 \text{ has } A'BC \text{ in } [x], & ABC \text{ in } [x'], \end{array}$$

wherein $A'TA_1'$, $A_1' \equiv (2, 1, 1)$.

The images of A' are thus arranged in pairs, $A'T^kA_k'$, $A'T^{n-k}A_{n-k}'$ being associated. When $n = 2m$, in T^m the fundamental triangles must coincide, since $(T^m)^2 = 1$, or $T^m = T^{-m}$.

8. For $n = 4$, $a = 2$, $m_1 = 1 + i$, $m_2 = 1 - i$. There are two pencils of cubics which remain invariant. The first is

$$k_1(x_1 - m_1x_3)(x_1 - m_2x_2)(x_2 - x_3) + k_2[(1 - m_2)x_1(x_2 - x_3) + (m_1 - m_2)\, x_3(x_1 - x_2)] \cdot l = 0,$$

wherein

$$l = m_1m_2(x_2 - x_3) + 2(m_1x_3 - m_2x_2) + x_1(m_2 - m_1) = 0.$$

The other is obtained from it by interchanging m_1, m_2. The first is composed of cubics having a node at D_3. The second is composed of harmonic curves.

9. When n is a prime greater than 3, the associated value of a is irrational.

10. The equations of the transformation will now be obtained from a different point of view, without any use of the preceding method. Since the pencils of straight lines through B and C are invariant, the transformation can be generated most easily by the Seydewitz method.*
The line $x_3 = 0$ goes over into $x_1' = 0$, $x_1 = 0$ into $x_1' - x_3' = 0$. The line $x_1y_3 - x_3y_1 = 0$ through the point (y) goes into

$$x_1'(ky_3 - y_1) = ky_3x_3',$$

k being an undefined parameter.

* Darstellung der geometrischen Verwandtschaften . . . , Archiv der Mathematik und Physik, vol. 7 (1846), pp. 113–148.

The values of m_1, m_2 in the equations of the invariant lines of the pencil

$$x_1 - m_1x_3 = 0, \quad x_1 - m_2x_3 = 0$$

are roots of the equation

$$m^2 - km + k = 0.$$

The characteristic anharmonic ratio of this linear transformation is m_1/m_2. If the period is n, then

$$m_1{}^n = m_2{}^n.$$

Since

$$m_1 + m_2 = k, \quad m_1m_2 = k, \quad m_1 \neq m_2,$$

we have an equation of order $E[(n-1)/2]$ to determine k, $E(s)$ being the largest integer not greater than s. It is exactly the preceding equation in a.

The same equations will hold for C if x_3 is replaced by x_2 and k by l. The period of this pencil may be n or any other positive integer. The line $x_1y_2 - x_2y_1 = 0$ joining (y) to C goes into $x_1(ly_2 - y_1) = y_2x_2$, hence (y'), the image of (y), is defined by the intersection of the lines

$$x_1(ly_2 - y_1) = ly_2x_2, \quad x_1(ky_3 - y_1) = ky_3x_3.$$

Replacing (y), (y') by (x), (x'), the equations of the transformation become

$$\begin{array}{llll} & \sigma x_1' = klx_2x_3, & & \rho x_1 = kl(x_1' - x_2')(x_1' - x_3'), \\ S: & \sigma x_2' = kx_2(lx_3 - x_1), & S^{-1}: & \rho x_2 = lx_1'(x_1' - x_2'), \\ & \sigma x_3' = lx_3(kx_2 - x_1), & & \rho x_3 = kx_1'(x_1' - x_3'). \end{array}$$

If $kl = a$, $l = a'$, $k = a''$, S is the product of T and the harmonic homology H: $x_1 = \sigma x_1'$, $x_2 = \sigma x_3'$, $x_3 = \sigma x_2'$, operating first with T and then with H. $S = TH$.

11. If $k = l$, the new equations of T reduce to the form previously derived. If $k \neq l$, the successive images of A' are not all on the line $x_2 - x_3 = 0$, nor do the images of a point P all lie on a conic of the pencil BCD_1D_2. The pencil $B(PP'D_1D_2)$ has the anharmonic ratio $m_1 : m_2$, while the pencil $C(PP'D_1D_2)$ has the ratio $m_1' : m_2'$. Even if the linear transformations of the pencils B and C have the same period $n (n > 3)$, k, l may be taken as different roots of the same equation. When n is prime, there are therefore $[(n-1)/2]^2$ different transformations of period n, but of these only $[(n-1)/2]$ have the images of a point all lie on a conic.

12. In Kantor's treatment, which is purely synthetic, the case $k \neq l$ is not considered. In consequence, all the finite groups generated by T and any other finite groups are omitted from his list, and also from that of Wiman, although the methods used by the latter are correct.

13. Transformations of the form T, $(k \neq l)$, cannot be regarded as the

plane depiction of a linear transformation in space under which a quadric surface remains invariant.

14. The remaining cases can all be obtained from T by first performing a cyclic linear transformation.

If $B' \equiv C$, $C' \equiv B$, first apply the harmonic homology $\sigma x_1 = a'a''x_1'$, $\sigma x_2 = a'^2 x_3'$, $\sigma x_3 = a''^2 x_2'$, then T.

If $A' \equiv B$, $C' \equiv C$, apply

$$\sigma x_1 = a^2 x_2', \quad \sigma x_2 = a'^2 x_1, \quad \sigma x_3 = aa'x_3'.$$

If $A' \equiv C$, $B' \equiv B$, apply

$$\sigma x_1 = a^2 x_3', \quad \sigma x_2 = aa''x_2', \quad \sigma x_3 = a''^2 x_3'.$$

If $A' \equiv B$, $B' \equiv C$, apply

$$\sigma x_1 = a^2 a' x_3', \quad \sigma x_2 = a'^2 a'' x_1', \quad \sigma x_3 = aa''^2 x_2'.$$

If $A' \equiv C$, $C' \equiv B$, apply

$$\sigma x_1 = a^2 a'' x_2', \quad \sigma x_2 = aa'^2 x_3', \quad \sigma x_3 = a'a''^2 x_1'.$$

The last two linear transformations are cyclic of order 3.

15. When two of the fundamental points approach coincidence, the same thing happens to the inverse, and we have a quadratic transformation of the second kind.* Let the images of the straight lines of the plane be a net of conics touching a line p at the point P, and also passing through another point Q. In order that each power of this transformation be quadratic, the inverse system of conics must have two of these basis points in common, that is, they must either pass through P and Q or touch p at P. In the former case they can have a common tangent p' at P $(p' \neq p)$, or all touch a common line q at Q. In the second case they also have another basis point R. These three cases will be considered in turn.

16. Let the inverse system pass through P and touch q at Q. Put

$$p \equiv x_3 = 0, \quad q \equiv x_2 = 0, \quad PQ \equiv x_1 = 0.$$

A simple calculation shows that the only possible form is

$$\begin{aligned} \sigma x_1' &= ax_1x_3, & \rho x_1 &= acx_1'x_2', \\ \sigma x_2' &= bx_1^2, & \rho x_2 &= a^2x_2'x_3', \\ \sigma x_3' &= cx_2x_3, & \rho x_3 &= bcx_1'^2. \end{aligned}$$

It is of period four, independent of a, b, c; its square is the involutorial quadratic transformation of the first kind.

* Doehlemann, l. c., p. 30; Scott, l. c., p. 222.

17. Let the inverse system touch p' at P and pass through Q. The equations become

$$\begin{aligned} \sigma x_1' &= a x_1 x_3, & \rho x_1 &= c x_1'(a x_3' - k x_1'), \\ \sigma x_2' &= c x_2 x_3, & \rho x_2 &= a x_2'(a x_3' - k x_1'), \\ \sigma x_3' &= d x_1^2 + k x_1 x_3, & \rho x_3 &= c d x_1'^2. \end{aligned}$$

The two projectivities are defined by

$$m_1' = \frac{a}{c} m_1, \quad m_2' = \frac{d + k m_2}{a m_2},$$

wherein

$$m_1 x_2 = x_1, \quad m_2 x_1 = x_3.$$

In this case transformations of any period greater than 2 may appear. The common tangent is $p' \equiv a x_3' - k x_1'$.

18. Let the inverse system touch p at P and pass through R. The general form of the transformation is

$$\begin{aligned} \sigma x_1' &= a x_1 x_3 + b x_1^2, \\ \sigma x_2' &= c x_2 x_3, \\ \sigma x_3' &= d x_1^2. \end{aligned}$$

the new basis point being $R \equiv (b, 0, d)$.

From the equations we have the two projectivities

$$\text{(1)} \qquad \frac{d x_1' - b x_3'}{a x_3'} = \frac{x_3}{x_1},$$

$$\text{(2)} \qquad \frac{a x_2'}{d x_1' - b x_3'} = \frac{c x_2}{d x_1}.$$

In the first, the two pencils are concentric, the invariant lines are $x_3 = m x_1$, wherein m is defined by the quadratic equation

$$a m^2 + b m - d = 0.$$

The second is a perspectivity between the lines of two non-concentric pencils, the axis passing through $P \equiv (0, 1, 0)$. By means of the figure suggested by (1) and (2) it is easily seen that *this transformation cannot be periodic for any values of a, b, c, d, except when $b = 0$*, in which case it does not belong to this type.

19. In case all three basis points approach coincidence the general form becomes *

* Doehlemann, l. c., p. 32; Scott, l. c., p. 222.

$$\rho x_1' = a(x_1x_2 - mx_3^2),$$
$$\rho x_2' = bx_2^2,$$
$$\rho x_3' = cx_2x_3.$$

The inverse is already in the proper form. By repeating the transformation the law of composition of the coefficients is at once seen. The transformation will be periodic, of period n, if

$$b = 1, \quad a^n = c^n = (ac)^n = 1, \quad a \neq 1, \quad c \neq 1.$$

There are as many types as there are sets of values of a and c satisfying these conditions. For $n > 5$ a point and its successive images will lie on a conic when and only when $a = c$. Neither Kantor nor Wiman considers any of the cases in which fundamental points approach coincidence.

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